POLYNOMIAL SPLITTING MEASURES AND COHOMOLOGY OF THE PURE BRAID GROUP

TREVOR HYDE AND JEFFREY C. LAGARIAS

ABSTRACT. We study for each n a one-parameter family of complex-valued measures on the symmetric group S_n, which interpolate the probability of a monic, degree n, square-free polynomial in $\mathbb{F}_q[x]$ having a given factorization type. For a fixed factorization type, indexed by a partition λ of n, the measure is known to be a Laurent polynomial. We express the coefficients of this polynomial in terms of characters associated to S_n-subrepresentations of the cohomology of the pure braid group $H^\bullet(P_n, \mathbb{Q})$. We deduce that the splitting measures for all parameter values $z = -\frac{1}{m}$ $\left(\text{resp. } z = \frac{1}{m}\right)$, after rescaling, are characters of S_n-representations (resp. virtual S_n-representations.)

1. INTRODUCTION

The purpose of this paper is to study for each $n \geq 1$ a one-parameter family of complex-valued measures on the symmetric group S_n arising from a problem in number theory, and to exhibit an explicit representation-theoretic connection between these measures and the characters of the natural S_n-action on the rational cohomology of the pure braid group P_n.

This family of measures, denoted $\nu^*_{n,z}$, was introduced by the second author and B. Weiss in [13], where they were called *z-splitting measures*, with parameter $z \in \mathbb{C}$. The measures interpolate from prime power values $z = q$ the probability of a monic, degree n, square-free polynomial in $\mathbb{F}_q[x]$ having a given factorization type. Square-free factorization types are indexed by partitions λ of n specifying the degrees of the irreducible factors. Each partition λ of n corresponds to a conjugacy class C_λ of the symmetric group S_n; distributing the probability of a factorization of type λ equally across the elements of C_λ defines a probability measure on S_n. A key property of the resulting probabilities is that for a fixed partition λ, their values are described by a rational function in the size of the field $\mathbb{F}_q$ as q varies. This permits interpolation from q to a parameter $z \in \mathbb{P}^1(\mathbb{C})$, from which one obtains the family of complex-valued measures $\nu^*_{n,z}$ on S_n given in Definition 2.3 below.

On the number theory side, these measures connect with problems on the splitting of ideals in S_n-number fields, which are degree n number fields formed by adjoining a root of a degree n polynomial over $\mathbb{Z}[x]$ whose splitting splitting field has Galois group S_n. The paper [13, Theorem 2.6] observed that for primes $p < n$

Date: April 18, 2016.
1991 *Mathematics Subject Classification.* Primary 11R09; Secondary 11R32, 12E20, 12E25.
Work of the second author was partially supported by NSF grant DMS-1401224.

these measures vanish on certain conjugacy classes, corrensponding to the phenomenon of essential discriminant divisors of polynomials having Galois group S_n, first noted by Dedekind [7] in 1878. These measures converge to the uniform measure on the symmetric group as $z = p \to \infty$, and in this limit agree with a conjecture of Bhargava [2, Conjecture 1.3] on the distribution of splitting types of the prime p in S_n-extensions of discriminant $|D| \leq B$ as the bound $B \to \infty$, conditioned on $(D, p) = 1$.

The second author [12] subsequently studied these measures at the special value $z = 1$ as representing splitting probabilities for polynomials over the (hypothetical) "field with one element $\mathbb{F}_1$." The splitting measures at $z = 1$ are signed measures for $n \geq 3$, unlike all other nonzero integral values of z. The 1-splitting measures are supported on a small set of conjugacy classes, the Springer regular elements of S_n (those conjugacy classes C_λ for which λ has a rectangular Young diagram or a rectangle plus a single box.) Viewed as class functions on S_n, rather than as measures, they were found to have a representation-theoretic interpretation: after rescaling by $n!$, they are virtual characters of S_n corresponding to explicitly determined representations. As n varies, their values on conjugacy classes were observed to have arithmetic properties compatible with the multiplicative structure of n; letting $n = \prod_p p^{e_p}$ be the canonical prime factorization of n, the measure's value on each conjugacy class factors corresponding to values on certain conjugacy classes of the smaller symmetric groups $S_{p^{e_p}}$. That paper also showed the rescaled z-splitting measures at $z = -1$ have a representation-theoretic interpretation.

These observations motivate further investigation of these measures, to locate a source for the connection with the representation theory of S_n. In this paper we find a representation-theoretic structure that extends to the entire family of z-splitting measures. Our starting point is the observation[1] that for a fixed conjugacy class the z-splitting measures are Laurent polynomials in z, cf. They have degree at most $n - 1$, so may be written

$$\nu^*_{n,z}(C_\lambda) = \sum_{k=0}^{n-1} \alpha_n^k(C_\lambda)\left(\tfrac{1}{z}\right)^k,$$

with coefficients $\alpha_n^k(C_\lambda)$, where λ is a partition of n and each $\alpha_n^k(C_\lambda)$ is a rational number. We call the $\alpha_n^k(C_\lambda)$ *splitting measure coefficients.* The novel observation of this paper is that each separate splitting measure coefficient $\alpha_n^k(C_\lambda)$, viewed as a function of λ, is a rescaled version of the character χ_n^k of a certain S_n-subrepresentation A_n^k of the cohomology of the pure braid group $H^k(P_n, \mathbb{Q})$. The pure braid groups P_n and their cohomology are discussed in Section 4, and the subrepresentations A_n^k described. We deduce as a consequence that the rescaled z-splitting measure is a character of S_n at $z = -\frac{1}{m}$ and is a virtual character of S_n at $z = \frac{1}{m}$, for all integers $m \geq 1$ (Theorem 5.2). This extends the representation-theoretic connection of [12] for $z = \pm 1$ to the parameter values $z = \pm\frac{1}{m}$ for all $m \geq 1$.

[1]Their definition shows they are rational functions having no poles on $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, 1\}$, and [12, Lemma 2.5] observed they have no poles at $z = 1$, cf. Proposition 2.4.

1.1. **Results.** The *z-splitting measure* on a conjugacy class C_λ of S_n is the rational function of z

$$\nu^*_{n,z}(C_\lambda) := \frac{N_\lambda(z)}{z^n - z^{n-1}},$$

where $N_\lambda(z) \in \mathbb{Q}[z]$ denotes the *cycle polynomial* associated to a partition λ describing the cycle lengths of C_λ. Given $\lambda = \left(1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots n^{m_n(\lambda)}\right)$, the associated cycle polynomial is

$$N_\lambda(z) := \prod_{j \geq 1} \binom{M_j(z)}{m_j(\lambda)}, \tag{1.1}$$

where $M_j(z)$ denotes the jth necklace polynomial. The *necklace polynomial* $M_j(z)$ of order j is given by

$$M_j(z) := \frac{1}{j} \sum_{d|j} \mu(d) z^{j/d}.$$

where $\mu(d)$ is the Möbius function.

To avoid confusion we make a remark on normalizations. Given a class function f on S_n we write $f(C_\lambda)$ for the sum of the values of f on C_λ and $f(\lambda)$ for the value $f(g)$ taken at one element $g \in C_\lambda$; the latter notation is standard for characters. Thus $\nu^*_{n,z}(C_\lambda) = |C_\lambda| \nu^*_{n,z}(\lambda)$. With this in mind, in Section 7.1 we introduce the *normalized splitting measure* ν_w, defined uniformly on partitions of any size, by

$$\nu_w(\lambda) := \frac{n!}{|C_\lambda|} \frac{w^n N_\lambda(w^{-1})}{1 - w}.$$

Each $\nu_w(\lambda)$ is a polynomial in w with integer coefficients and constant term 1.

In Section 3 we express the family of cycle polynomials $N_\lambda(z)$ in terms of characters of the cohomology of the pure braid group P_n viewed as an S_n-representation.

Theorem 1.1 (Character interpretation of cycle polynomial coefficients)**.** *Let λ be a partition of n and $N_\lambda(z)$ be a cycle polynomial. Then*

$$N_\lambda(z) = \frac{|C_\lambda|}{n!} \sum_{k=0}^{n} (-1)^k h_n^k(\lambda) z^{n-k}.$$

where h_n^k is the character of the kth cohomology of the pure braid group $H^k(P_n, \mathbb{Q})$, viewed as an S_n-representation.

Theorem 1.1 is proven in Section 3 (as Theorem 3.2) using the twisted Grothendieck-Lefschetz formula of Church, Ellenberg, and Farb [4, Prop. 4.1]. Comparing this formula for $N_\lambda(z)$ with (1.1), we deduce many properties of the characters h_n^k in an elementary fashion.

In Section 4 we review the pure braid group and its cohomology, and derive an exact sequence producing S_n-subrepresentations A_n^k of $H^k(P_n, \mathbb{Q})$. The main result of this paper, given in the Section 5, is an expression of the z-splitting measures $\nu^*_{n,z}$ in terms of the characters χ_n^k of A_n^k.

Theorem 1.2 (Character interpretation of splitting measure coefficients)**.** *For each $n \geq 1$ and $0 \leq k \leq n-1$ there is an S_n-subrepresentation A_n^k of $H^k(P_n, \mathbb{Q})$ (constructed explicitly in Proposition 4.2) with character χ_n^k such that for each partition λ of n,*

$$\nu_{n,z}^*(C_\lambda) = \frac{|C_\lambda|}{n!} \sum_{k=0}^{n-1} (-1)^k \chi_n^k(\lambda) \left(\tfrac{1}{z}\right)^k.$$

Thus the splitting measure coefficient $\alpha_n^k(C_\lambda)$ is

$$\alpha_n^k(C_\lambda) = |C_\lambda| \, \alpha_n^k(\lambda) = (-1)^k \frac{|C_\lambda|}{n!} \chi_n^k(\lambda).$$

We obtain in Section 5.2 a corollary of this result: For $z = -\frac{1}{m}$ with $m \geq 1$, the rescaled splitting measure itself $\frac{n!}{|C_\lambda|}\nu_{n,z}^k(C_\lambda)$ is the character of an S_n-representation, and when $z = \frac{1}{m}$ it is the character of a virtual S_n-representation.

In Section 5.3 we use Theorem 1.2 and values of the (-1)-splitting measure computed in [12] to prove the following relation between the S_n-representation structure of the pure braid group cohomology and the regular representation $\mathbb{Q}[S_n]$.

Theorem 1.3. *Let $\mathbf{1}_n$, $\mathbf{Sgn}_n$, and $\mathbb{Q}[S_n]$ be the trivial, sign, and regular representations respectively of S_n respectively. Then there is an isomorphism of S_n-representations,*

$$\bigoplus_{k=0}^{n} H^k(P_n, \mathbb{Q}) \otimes \mathbf{Sgn}_n^{\otimes k} \cong \mathbb{Q}[S_n].$$

Here $\mathbf{Sgn}_n^{\otimes k} \cong \mathbf{1}_n$ or $\mathbf{Sgn}_n$ according to whether k is even or odd.

The cohomology groups $H^k(P_n, \mathbb{Q})$ are well studied, and were described as S_n-representations in 1986 by Lehrer and Solomon [14, Theorem 4.5] in terms of induced representations $\mathrm{Ind}_Z^{S_n}(\xi_\lambda)$ for explicitly defined linear characters ξ_λ on the centralizers $Z(C_\lambda)$ of conjugacy classes C_λ having $n-k$ cycles. However Theorem 1.3 seems not to have been explicitly noted before.

In Section 6 we study representation stability properties of the splitting measure coefficients. For fixed k and varying n, the sequence of S_n-representations $H^k(P_n, \mathbb{Q})$ exhibits *representation stability* in the sense of Church and Farb [6] (see [4], [5]). We show in Proposition 6.2 that the representations A_n^k are isomorphic to others appearing in the literature known to exhibit representation stability. Hersh and Reiner [11] determine their precise rate of stabilization, yielding the following result.

Theorem 1.4 (Representation stability for A_n^k)**.** *For each fixed $k \geq 1$, the sequence of S_n-representations A_n^k with characters χ_n^k are representation stable, and stabilize sharply at $n = 3k+1$.*

In Section 7 we demonstrate a further manifestation of representation stability, visible in the behavior of the normalized splitting measures ν_w. We show that ν_w is *continuous with respect to stable limits* in the following sense: given partitions

$\lambda^{(1)}, \lambda^{(2)}$ we say $\lambda^{(1)} \subseteq \lambda^{(2)}$ if the Young diagram of $\lambda^{(1)}$ fits inside that of $\lambda^{(2)}$. A *nested sequence* of partitions $\lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \lambda^{(3)} \subseteq \dots$ is any infinite chain of partitions. We call a nested sequence of partitions $\big(\lambda^{(k)}\big)$ a *stable sequence* if the lengths $\ell\big(\lambda^{(k)}\big)$ are bounded. Given a stable sequence $\big(\lambda^{(k)}\big)$, let $s \geq 1$ denote the minimal s such that $\big(\lambda_s^{(k)}\big)$ is eventually constant, and define its *stable limit* to be the partition

$$\operatorname*{stablim}_{k\to\infty} \lambda^{(k)} := \lim_{k\to\infty} \big[\lambda_s^{(k)}, \lambda_{s+1}^{(k)}, \lambda_{s+2}^{(k)}, \dots\big],$$

where the limit on the right hand side is taken componentwise.

Theorem 1.5 (Continuity with respect to stable limits)**.** *Suppose* $\big(\lambda^{(k)}\big)$ *is a stable sequence of partitions, then*

$$\lim_{k\to\infty} \nu_w\big(\lambda^{(k)}\big) = \nu_w\big(\operatorname*{stablim}_{k\to\infty} \lambda^{(k)}\big),$$

where the limit on the left hand side is taken coefficientwise in $\mathbb{Z}[w]$ *(i.e. it is a limit viewed in the formal power series ring* $\mathbb{Z}[[w]]$*) and the limit on the right hand side is the stable limit of a sequence of partitions.*

Theorem 1.5 may be interpreted as saying the values of $\chi_n^k(\lambda)$ on a stable sequence of partitions are eventually constant, hence independent of n for large enough n. The convergence rate is related to the stabilization rate of the representations A_n^k.

Finally we note that the representation-theoretic interpretation given here raises new questions concerning the splitting measures. For example, is there some direct interpretation of the splitting measures at the parameter values $z = \pm\frac{1}{m}$ that explains their interpretation as scaled characters of representations (resp. virtual representations)?

1.2. **Plan of the Paper.** In Section 2 we recall properties of the z-splitting measures from [13]. In Section 3 we use the twisted Grothendieck-Lefschetz formula to relate the coefficients of cycle polynomials to the characters of the S_n-representations $H^k(P_n, \mathbb{Q})$. In Section 4 we discuss the cohomology $H^k(P_n, \mathbb{Q})$ of the pure braid group P_n, and derives an exact sequence leading to the construction of the S_n-representations A_n^k. In Section 5 we express the splitting measure coefficients $\alpha_n^k(C_\lambda)$ in terms of the character χ_n^k of the representation A_n^k. In Section 6 we discuss representation stability and connect the S_n-representations A_n^k with others in the literature. In Section 7 we introduce the normalized splitting measure and prove a continuity property with respect to stable limits.

1.3. **Notation.**

(1) $q = p^f$ always denotes a prime power.
(2) The set of monic, degree n, square-free polynomials in $\mathbb{F}_q[x]$ is denoted $\mathrm{Conf}_n(\mathbb{F}_q)$.
(3) We write partitions either as $\lambda = \big[\lambda_1, \lambda_2, \cdots, \lambda_\ell\big]$, with parts $\lambda_1 \geq \lambda_2 \geq \cdots$ eventually 0, or as $\lambda = (1^{m_1} 2^{m_2} \cdots)$ where $m_j = m_j(\lambda)$ is the number of parts of λ of size j. The length of λ is $\ell(\lambda) = \max\{r : \lambda_r \geq 1\}$, the size of λ is $|\lambda| = \sum_i \lambda_i = \sum_j j m_j$, and λ_i is the ith largest part of λ.

(4) Each partition λ of n corresponds to a conjugacy class C_λ of S_n given by the common cycle structure of the elements in C_λ. We let Z_λ denote the centralizer of C_λ in S_n. The size of the centralizer and conjugacy class are

$$z_\lambda := |Z_\lambda| = \prod_{j\geq 1} j^{m_j(\lambda)} m_j(\lambda)! \qquad c_\lambda := |C_\lambda| = \frac{n!}{z_\lambda}$$

respectively. Note that $c_\lambda z_\lambda = n!$.

(5) Following [19], we let $\mathrm{Par}(n)$ denote the set of partitions of n and $\mathrm{Par} = \bigcup_n \mathrm{Par}(n)$ the set of all partitions.

2. Splitting Measures

We review the splitting measures introduced in [13], summarize their properties, and introduce the normalized splitting measures.

2.1. Necklace polynomials and cycle polynomials.

Definition 2.1. For $j \geq 1$, the *jth necklace polynomial* $M_j(z) \in \frac{1}{j}\mathbb{Z}[z]$ is

$$M_j(z) := \frac{1}{j}\sum_{d|j} \mu(d) z^{j/d},$$

where $\mu(d)$ is the Möbius function.

Moreau [16] noted in 1872 that for all integers $m \geq 1$, $M_j(m)$ is the number of distinct necklaces having j beads drawn from a set of m colors, up to cyclic permutation. This fact motivated Metropolis and Rota [15] to name them *necklace polynomials.* Relevant to the present paper, $M_j(q)$ is the number of monic, degree j, irreducible polynomials in $\mathbb{F}_q[x]$ [17, Prop. 2.1]. The factorization type of a polynomial $f \in \mathrm{Conf}_n(\mathbb{F}_q)$ is the collection of degrees of its irreducible factors, which we write $[f]$.

Definition 2.2. Given a partition λ of n, the *cycle polynomial* $N_\lambda(z) \in \frac{1}{z_\lambda}\mathbb{Z}[z]$ is

$$N_\lambda(z) := \prod_{j\geq 1} \binom{M_j(z)}{m_j(\lambda)},$$

where $\binom{\alpha}{m}$ is the usual extension of a binomial coefficient,

$$\binom{\alpha}{m} := \frac{1}{m!}\prod_{k=0}^{m-1}(\alpha - k).$$

The cycle polynomial $N_\lambda(z)$ has degree $n = |\lambda|$ and is integer valued for $z \in \mathbb{Z}$. The number of $f \in \mathrm{Conf}_n(\mathbb{F}_q)$ with $[f] = \lambda$ is $N_\lambda(q)$ (see [13, Sect. 4].)

2.2. z**-splitting measures.** If λ a partition of n, then the probability of a uniformly chosen $f \in \mathrm{Conf}_n(\mathbb{F}_q)$ having factorization type λ is

$$\mathrm{Prob}\{f \in \mathrm{Conf}_n(\mathbb{F}_q) : [f] = \lambda\} = \frac{N_\lambda(q)}{|\mathrm{Conf}_n(\mathbb{F}_q)|}.$$

When $n = 1$, $|\mathrm{Conf}_n(\mathbb{F}_q)| = q$ and for $n \geq 2$ we have $|\mathrm{Conf}_n(\mathbb{F}_q)| = q^n - q^{n-1}$. (See [17, Prop. 2.3] for a proof via generating functions. A proof due to Zieve appears in [22, Lem. 4.1].) Hence, the probability is a rational function in q. Replacing q by a complex-valued parameter z yields the z-splitting measure.

Definition 2.3. *For $n \geq 2$ the* z-splitting measure $\nu^*_{n,z}(C_\lambda) \in \mathbb{Q}(z)$ *is given by*

$$\nu^*_{n,z}(C_\lambda) := \frac{N_\lambda(z)}{z^n - z^{n-1}}.$$

Proposition 2.4. *For each partition λ of $n \geq 1$, the rational function $\nu^*_{n,z}(C_\lambda)$ is a polynomial in $\frac{1}{z}$ of degree at most $n-1$. Thus it may be written as*

$$\nu^*_{n,z}(C_\lambda) = \sum_{k=0}^{n-1} \alpha_n^k(C_\lambda)\left(\tfrac{1}{z}\right)^k.$$

The function $\nu^*_{1,z}(C_1) = 1$ is independent of z.

Proof. The case $n = 1$ is clear. For $n \geq 2$ we have $N_\lambda(1) = 0$ by [12, Lemma 2.5], whence $\frac{N_\lambda(z)}{z-1}$ is a polynomial of degree at most $n-1$ in z. Therefore,

$$\nu^*_{n,z}(C_\lambda) = \frac{N_\lambda(z)}{z^n - z^{n-1}} = \frac{1}{z^{n-1}}\left(\frac{N_\lambda(z)}{z-1}\right)$$

is a polynomial in $\frac{1}{z}$ of degree at most $n-1$. ☐

For $n \geq 2$ the Laurent polynomial $\nu^*_{n,z}(C_\lambda)$ is of degree at most $n-2$ since $z \mid N_\lambda(z)$ ([13, Lemma 4.3]); that is, $\alpha_n^{n-1}(C_\lambda) = 0$. Tables 1 and 2 give $\nu^*_{n,z}(C_\lambda)$, exhibiting the splitting measure coefficients $\alpha_n^k(C_\lambda)$ for $n = 4$ and $n = 5$.

λ	c_λ	z_λ	$\nu^*_{4,z}(C_\lambda)$
$[1,1,1,1]$	1	24	$\frac{1}{24}\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)$
$[2,1,1]$	6	4	$\frac{1}{4}\left(1 - \frac{1}{z}\right)$
$[2,2]$	3	8	$\frac{1}{8}\left(1 - \frac{1}{z} - \frac{2}{z^2}\right)$
$[3,1]$	8	3	$\frac{1}{3}\left(1 + \frac{1}{z}\right)$
$[4]$	6	4	$\frac{1}{4}\left(1 + \frac{1}{z}\right)$

TABLE 1. Values of the z-splitting measures $\nu^*_{4,z}(C_\lambda)$ on partitions λ of $n = 4$.

λ	c_λ	z_λ	$\nu^*_{5,z}(C_\lambda)$
$[1,1,1,1,1]$	1	120	$\frac{1}{120}\left(1-\frac{9}{z}+\frac{26}{z^2}-\frac{24}{z^3}\right)$
$[2,1,1,1]$	10	12	$\frac{1}{12}\left(1-\frac{3}{z}+\frac{2}{z^2}\right)$
$[2,2,1]$	15	8	$\frac{1}{8}\left(1-\frac{1}{z}-\frac{2}{z^2}\right)$
$[3,1,1]$	20	6	$\frac{1}{6}\left(1-\frac{1}{z^2}\right)$
$[3,2]$	20	6	$\frac{1}{6}\left(1-\frac{1}{z^2}\right)$
$[4,1]$	30	4	$\frac{1}{4}\left(1+\frac{1}{z}\right)$
$[5]$	24	5	$\frac{1}{5}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}\right)$

TABLE 2. Values of the z-splitting measures $\nu^*_{5,z}(C_\lambda)$ on partitions λ of $n=5$.

3. INTERPRETATION OF CYCLE POLYNOMIAL COEFFICIENTS

The cycle polynomials $N_\lambda(z) \in \frac{1}{z_\lambda}\mathbb{Z}[z]$ were defined for each partition λ of n in Section 2.1. We express the coefficients of $N_\lambda(z)$ in terms of characters h_n^k of the cohomology of the pure braid group P_n viewed as an S_n-representation. The connection is made through the twisted Grothendieck-Lefschetz formula of Church, Ellenberg, and Farb [4]. Using explicit formulas for the cycle polynomials we obtain constraints on the support of h_n^k. We compute $h_n^k(\lambda)$ for all n in several examples, fixing either the dimension k or the partition λ and varying the other.

3.1. **Cohomology of the pure braid group.** Given a set X of n distinct points in 3-dimensional affine space, the *braid group* B_n consists of homotopy classes of simple, non-intersecting paths beginning and terminating in X, with concatenation as the group operation. Each element of B_n determines a permutation of X, giving a short exact sequence of groups

$$0 \to P_n \to B_n \xrightarrow{\pi} S_n \to 0.$$

Then $P_n := \ker \pi$ is called the *pure braid group*. P_n consists of homotopy classes of simple, non-intersecting *loops* based in X. The action of S_n on X induces an action on P_n by permuting the loops. Thus, for each k, the kth group cohomology $H^k(P_n, \mathbb{Q})$ is an S_n-representation whose character we denote by h_n^k.

3.2. **Twisted Grothendieck-Lefschetz formula.** A *character polynomial* is a polynomial $P(x) \in \mathbb{Q}[x_j : j \geq 1]$. Character polynomials induce functions $P : \mathrm{Par} \to \mathbb{Q}$ by

$$P(\lambda) := P\big(m_1(\lambda), m_2(\lambda), \ldots\big),$$

noting that $m_i(\lambda) = 0$ for all but finitely many i. For $f \in \mathrm{Conf}_n(\mathbb{F}_q)$ we let $P(f) := P([f])$. Given two $\mathbb{Q}$-valued functions F and G defined on S_n let

$$\langle F, G\rangle := \frac{1}{n!}\sum_{\sigma \in S_n} F(\sigma)G(\sigma).$$

The following Theorem is due to Church, Ellenberg, and Farb [4, Prop. 4.1].

Theorem 3.1 (Twisted Grothendieck-Lefschetz for PConf_n)**.** *Given a prime power q, an integer $n \geq 1$, and a character polynomial P, we have*

$$\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} P(f) = \sum_{k=0}^{n} (-1)^k \left\langle P, h_n^k \right\rangle q^{n-k}, \tag{3.1}$$

where h_n^k is the character of the cohomology of the pure braid group $H^k(P_n, \mathbb{Q})$.

The classic *Lefschetz trace formula* counts the fixed points of an endomorphism f on a compact manifold M by the trace of the induced map on the singular cohomology of M. One may interpret the $\overline{\mathbb{F}}_q$ points on an algebraic variety V defined over $\mathbb{F}_q$ as the fixed points of the *geometric Frobenius endomorphism* of V. Using the machinery of ℓ-adic étale cohomology, Grothendieck [10] generalized Lefschetz's formula to count the number of points in $V(\mathbb{F}_q)$ by the trace of Frobenius on the étale cohomology of V. For nice varieties V defined over $\mathbb{Z}$, there are comparison theorems relating the étale cohomology of $V(\overline{\mathbb{F}}_q)$ to the singular cohomology of $V(\mathbb{C})$. This connects the topology of a complex manifold to point counts of a variety over a finite field.

Church, Ellenberg, and Farb [4] build upon Grothendieck's extension of the Lefschetz formula to relate point counts on natural subsets of $\mathrm{Conf}_n(\mathbb{F}_q)$ to the singular cohomology of the covering space $\mathrm{PConf}_n(\mathbb{C}) \to \mathrm{Conf}_n(\mathbb{C})$. $\mathrm{PConf}_n(\mathbb{C})$ is the space of n distinct, labelled points in $\mathbb{C}$. The space $\mathrm{PConf}_n(\mathbb{C})$ has fundamental group P_n, the pure braid group, and is a $K(\pi, 1)$ for this group. Hence, the singular cohomology of $\mathrm{PConf}_n(\mathbb{C})$ is the same as the group cohomology of P_n. This provides the connection between $\mathrm{Conf}_n(\mathbb{F}_q)$ on the left hand side of (3.1) and the character of the pure braid group cohomology on the right hand side.

3.3. **Cycle polynomials and pure braid group cohomology.** Theorem 3.2 expresses $N_\lambda(z)$ in terms of the characters h_n^k using Theorem 3.1.

Theorem 3.2. *Let λ be a partition of n, then*

$$N_\lambda(z) = \frac{1}{z_\lambda} \sum_{k=0}^{n} (-1)^k h_n^k(\lambda) z^{n-k},$$

where h_n^k is the character of the S_n-representation $H^k(P_n, \mathbb{Q})$.

Proof. Define the character polynomial $1_\lambda(x) \in \mathbb{Q}[x_j : j \geq 1]$ by

$$1_\lambda(x) = \prod_{j \geq 1} \binom{x_j}{m_j(\lambda)}.$$

Observe that for a partition $\mu \in \mathrm{Par}(n)$ we have

$$1_\lambda(\mu) = \begin{cases} 1 & \text{if } \mu = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$N_\lambda(q) = \sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} 1_\lambda(f).$$

On the other hand, by Theorem 3.1 we have

$$\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} 1_\lambda(f) = \sum_{k=0}^{n} (-1)^k \langle 1_\lambda, h_n^k \rangle q^{n-k}.$$

If $\sigma \in S_n$, let $[\sigma] \in \mathrm{Par}(n)$ be the partition given by the cycle lengths of σ. Thus,

$$\langle 1_\lambda, h_n^k \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} 1_\lambda(\sigma) h_n^k(\sigma) = \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ [\sigma]=\lambda}} h_n^k(\sigma) = \frac{c_\lambda}{n!} h_n^k(\lambda) = \frac{1}{z_\lambda} h_n^k(\lambda).$$

Therefore the identity

$$N_\lambda(q) = \frac{1}{z_\lambda} \sum_{k=0}^{n} (-1)^k h_n^k(\lambda) q^{n-k}$$

holds for all prime powers q, giving the identity as polynomials in $\mathbb{Q}[z]$. $\square$

Remark. A recent result of Chen [3, Thm. 1] also gives the identity in Theorem 3.2 by specializing at $t = 0$.

Recall that for a partition λ of n the cycle polynomial $N_\lambda(z)$ is defined by

$$N_\lambda(z) = \prod_{j \geq 1} \binom{M_j(z)}{m_j(\lambda)}, \tag{3.2}$$

where

$$M_j(z) = \frac{1}{j} \sum_{d \mid j} \mu(d) z^{j/d}$$

is the jth necklace polynomial.

Theorem 3.2 allows us to compute $h_n^k(\lambda)$ by expanding the explicit formula for $N_\lambda(z)$ and comparing coefficients. We illustrate this by deducing constraints on the support of h_n^k in Proposition 3.3 and computing values of $h_n^k(\lambda)$ in the examples of Sections 3.5 and 3.6.

3.4. **Support restrictions on characters h_n^k.** The character h_n^k is supported on partitions with at least one small part, while h_n^{n-k} is supported on partitions with at most k different parts (staircase partitions with at most k steps).

Proposition 3.3. *Let $0 \leq k \leq n$ and h_n^k be the character of the S_n-representation $H^k(P_n, \mathbb{Q})$, then*

(1) h_n^k is supported on partitions having at least one part of size at most $2k$. The value $h_n^k(\lambda)$ is determined by $m_j(\lambda)$ for $1 \leq j \leq 2k$.

(2) h_n^{n-k} is supported on partitions λ such that $m_j(\lambda) > 0$ for at most k distinct values of j.

Proof. (1) Theorem 3.2 implies $h_n^k(\lambda)$ is nonzero iff the coefficient of z^{n-k} in $N_\lambda(z)$ is nonzero. The degree of $M_j(z) - \frac{1}{j}z^j$ is at most $\lfloor j/2 \rfloor$. Hence if $j > 2k$, then the coefficient of z^{n-k} in $\binom{M_j(z)}{m_j(\lambda)}$ is zero. Thus the only j contributing to the coefficient of z^{n-k} in (3.2) are those with $1 \leq j \leq 2k$.

(2) Theorem 3.2 implies $h_n^{n-k}(\lambda)$ is nonzero iff the coefficient of z^k in $N_\lambda(z)$ is nonzero. If $m_j(\lambda) > 0$, then z divides $\binom{M_j(z)}{m_j(\lambda)}$. Hence if $m_j(\lambda) > 0$ for more than k values of j, then $h_n^{n-k}(\lambda) = 0$. $\square$

Remark. Property (1) is a manifestation of representation stability. A stronger property of h_n^k is that it is given by a character polynomial for $n \geq 3k+1$, see Example 3.7 and Section 6. The determination of these character polynomials remains an open question [9]. Proposition 3.3 bounds which m_j may occur in the character polynomial for h_n^k.

3.5. **Character values $h_n^k(\lambda)$ for fixed λ and varying k.** 5 We compute $h_n^k(\lambda)$ for fixed λ and varying k by expanding the cycle polynomial $N_\lambda(z)$.

Example 3.4 (Dimensions of cohomology)**.** The dimension of $H^k(P_n, \mathbb{Q})$ is the value of h_n^k at the identity element, corresponding to the partition (1^n). Since $M_1(z) = z$ and the centralizer of the identity in S_n has order $z_{(1^n)} = n!$. We have

$$N_{(1^n)}(z) = \binom{z}{n} = \frac{1}{n!}\prod_{i=0}^{n-1}(z-i) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ n-k \end{bmatrix} z^{n-k},$$

where $\left[\begin{smallmatrix} n \\ n-k \end{smallmatrix}\right]$ is an *unsigned Stirling number of the first kind*. Theorem 3.2 says

$$N_{(1^n)}(z) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k h_n^k\big((1^n)\big) z^{n-k}.$$

Comparing coefficients recovers the well-known formula for the dimension of the pure braid group cohomology:

$$\dim H^k(P_n, \mathbb{Q}) = h_n^k\big((1^n)\big) = \begin{bmatrix} n \\ n-k \end{bmatrix}.$$

This result was observed by Arnol'd [1]. These values are given in Table 3.

Example 3.5. The partition $\lambda = [n]$ corresponds to an n-cycle in S_n. The centralizer of an n-cycle has order $z_{[n]} = n$ and

$$N_{[n]}(z) = \binom{M_n(z)}{1} = M_n(z) = \frac{1}{n}\sum_{d|n}\mu(d) z^{n/d}. \tag{3.3}$$

Theorem 3.2 gives us

$$N_{[n]}(z) = \frac{1}{n}\sum_{k=0}^{n}(-1)^k h_n^k\big([n]\big) z^{n-k}. \tag{3.4}$$

$n \setminus k$	0	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0
3	1	3	2	0	0	0	0	0	0
4	1	6	11	6	0	0	0	0	0
5	1	10	35	50	24	0	0	0	0
6	1	15	85	225	274	120	0	0	0
7	1	21	175	735	1624	1764	720	0	0
8	1	28	322	1960	6769	13132	13068	5040	0
9	1	36	546	4536	22449	67284	118124	109584	40320

TABLE 3. Betti numbers of pure braid group cohomology $H^k(P_n, \mathbb{Q})$.

Comparing coefficients, we find that

$$h_n^{n-k}([n]) = \begin{cases} (-1)^{n-k}\mu(\frac{n}{k}) & \text{if } k \mid n\ , \\ \\ 0 & \text{if } k \nmid n\ . \end{cases}$$

Note that the coefficients of $N_{[n]}(z)$ are determined by the *multiplicative structure* of n in (3.3) and by the *additive structure* of n in (3.4).

3.6. **Character values $h_n^k(\lambda)$ for fixed k varying λ.** We now compute $h_n^k(\lambda)$ for fixed k and varying λ.

Example 3.6 (Computing h_n^0 and h_n^n)**.** The cases $k = 0$ and n are both constant: $h_n^0 = 1$ and $h_n^n = 0$. The leading coefficient of $N_\lambda(z)$ is $1/z_\lambda$, hence Theorem 3.2 tells us $h_n^0(\lambda) = 1$ for all λ. For $j \geq 1$, we have $z \mid M_j(z)$, from which it follows that $z \mid N_\lambda(z)$ for all partitions λ of $n \geq 1$. In other words, for all $m_j \geq 1$

$$\frac{1}{z_\lambda}(-1)^n h_n^n(\lambda) = N_\lambda(0) = 0.$$

Thus $h_n^n(\lambda) = 0$ for all λ, and $H^n(P_n, \mathbb{Q}) = 0$.

Example 3.7 (Computing h_n^1 and h_n^2)**.** Taking $\lambda = (1^{m_1}2^{m_2}\cdots)$, a careful analysis of the z^{n-1} and z^{n-2} coefficients in $N_\lambda(z)$ and Theorem 3.2 yields the following formulas

$$h_n^1(\lambda) = \binom{m_1}{2} + \binom{m_2}{1}$$
$$h_n^2(\lambda) = 2\binom{m_1}{3} + 3\binom{m_1}{4} + \binom{m_1}{2}\binom{m_2}{1} - \binom{m_2}{2} - \binom{m_3}{1} - \binom{m_4}{1},$$

where $m_j = m_j(\lambda)$. These formulas represent h_n^1 and h_n^2 as character polynomials, and they appear in [4, Lemma 4.8]. Note that $h_n^1(\lambda) = h_n^2(\lambda) = 0$ for partitions λ having all parts larger than 2 and 4 respectively, illustrating Proposition 3.3 (1).

Example 3.8 (Computing h_n^{n-1})**.** The z coefficient of $N_\lambda(z)$ determines the value of $h_n^{n-1}(\lambda)$. Since each j with $m_j(\lambda) > 0$ contributes a factor of z to $N_\lambda(z)$, h_n^{n-1}

is supported on partitions of the form $\lambda = (j^m)$. Note that the z coefficient of the necklace polynomial $M_j(z)$ is $\mu(j)/j$. Let $\lambda = (j^m)$, then the z coefficient of

$$N_\lambda(z) = \binom{M_j(z)}{m} = \frac{M_j(z)(M_j(z)-1)\cdots(M_j(z)-m+1)}{m!}$$

is $(-1)^{m-1}\frac{\mu(j)}{jm}$. Since $z_\lambda = j^m m!$, we conclude

$$h_n^{n-1}(\lambda) = \begin{cases} (-1)^{m-n}\mu(j)j^{m-1}(m-1)! & \text{if } \lambda = (j^m)\,, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.9 (Computing h_n^{n-2})**.** The z^2 coefficient of $N_\lambda(z)$ determines $h_n^{n-2}(\lambda)$. Proposition 3.3 (2) tells us that $h_n^{n-2}(\lambda) = 0$ when $m_j(\lambda) > 0$ for at least three j. We treat the two remaining cases $\lambda = (i^{m_i} j^{m_j})$ and $\lambda = (j^m)$ in turn. If $\lambda = (i^{m_i} j^{m_j})$, then the z coefficient of $\binom{M_i(z)}{m_i}$ is $(-1)^{m_i-1}\frac{\mu(i)}{im_i}$, and similarly for $\binom{M_j(z)}{m_j}$. We have $z_\lambda = (i^{m_i} m_i!)(j^{m_j} m_j!)$. Thus, by Theorem 3.2

$$\begin{aligned} h_n^{n-2}\big((i^{m_1} j^{m_j})\big) &= (-1)^{m_i+m_j-n} z_\lambda \frac{\mu(i)\mu(j)}{(im_i)(jm_j)} \\ &= (-1)^{m_i+m_j-n}\big(\mu(i)i^{m_i-1}(m_i-1)!\big)\big(\mu(j)j^{m_j-1}(m_j-1)!\big). \end{aligned}$$

If $\lambda = (j^m)$, then the z^2 coefficient of $N_\lambda(z)$ receives a contribution of $(-1)^{m-1}\frac{\mu(j/2)}{jm}$ from the quadratic term of $M_j(z)$ if j is even. The z coefficient of $\binom{M_j(z)}{m_j}/M_j(z)$ is

$$\frac{\mu(j)}{jm!}\left(\sum_{i=1}^{m-1}\frac{(-1)^{m-2}(m-1)!}{i}\right) = (-1)^m\frac{\mu(j)}{jm}H_{m-1},$$

where $H_{m-1} = \sum_{i=1}^{m-1}\frac{1}{i}$ denotes the $(m-1)$*-th harmonic number*. The z coefficient of $M_j(z)$ is $\frac{\mu(j)}{j}$. Using the convention that the Möbius function $\mu(\alpha)$ vanishes at non-integral α, we arrive at the following expression for $h_n^{n-2}(\lambda)$:

$$\begin{aligned} h_n^{n-2}\big((j^m)\big) &= z_\lambda(-1)^{m-n}\frac{\big(\mu(j)^2 H_{m-1} - \mu(\frac{j}{2})\big)}{jm} \\ &= (-1)^{m-n}\big(\mu(j)^2 H_{m-1} - \mu(\tfrac{j}{2})\big)j^{m-1}(m-1)!. \end{aligned}$$

4. An Exact Sequence of Pure Braid Group Cohomology

Arnol'd [1] gave the following presentation of the cohomology ring $H^\bullet(P_n,\mathbb{Q})$ of the pure braid group P_n as an S_n-algebra.

Theorem 4.1 (Arnol'd)**.** *There is an isomorphism of graded S_n-algebras*

$$H^\bullet(P_n,\mathbb{Q}) \cong \Lambda^\bullet[\omega_{i,j}]/\langle R_{i,j,k}\rangle,$$

where $1 \le i,j,k \le n$ are distinct, $\omega_{i,j} = \omega_{j,i}$ have degree 1, and

$$R_{i,j,k} = \omega_{i,j}\wedge\omega_{j,k} + \omega_{j,k}\wedge\omega_{k,i} + \omega_{k,i}\wedge\omega_{i,j}.$$

An element $\sigma \in S_n$ acts on $\omega_{i,j}$ by $\sigma \cdot \omega_{i,j} = \omega_{\sigma(i),\sigma(j)}$.

Proposition 4.2 below uses Theorem 4.1 to construct an exact sequence of the pure braid group cohomology. This allows us to define the sequence A_n^k of S_n-representations whose characters determine the splitting measure coefficients $\alpha_n^k(C_\lambda)$.

In what follows, we identify $H^\bullet(P_n, \mathbb{Q})$ with this presentation as a quotient of an exterior algebra. The ring $\Lambda^\bullet[\omega_{i,j}]/\langle R_{i,j,k}\rangle$ is an example of an *Orlik-Solomon algebra*, which arise as cohomology rings of complements of hyperplane arrangements (see Dimca and Yuzvinsky [8] and Yuzvinsky [23].)

4.1. **An exact sequence.** Let $\tau = \sum_{1 \le i < j \le n} \omega_{i,j} \in H^1(P_n, \mathbb{Q})$. The element τ generates a trivial S_n-subrepresentation of $H^1(P_n, \mathbb{Q})$. We define maps $d^k : H^k(P_n, \mathbb{Q}) \to H^{k+1}(P_n, \mathbb{Q})$ for each k by $\nu \mapsto \nu \wedge \tau$. This map is linear and S_n-equivariant, since

$$\sigma \cdot d^k(\nu) = \sigma \cdot (\nu \wedge \tau) = (\sigma \cdot \nu) \wedge (\sigma \cdot \tau) = (\sigma \cdot \nu) \wedge \tau = d^k(\sigma \cdot \nu).$$

From $d^{k+1} \circ d^k = 0$ we conclude that

$$0 \to H^0(P_n, \mathbb{Q}) \xrightarrow{d^0} H^1(P_n, \mathbb{Q}) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H^n(P_n, \mathbb{Q}) \xrightarrow{d^n} 0$$

is a chain complex of S_n-representations. It follows from the general theory of Orlik-Solomon algebras that the above sequence is exact [8, Thm. 5.2]. We include a proof in this case for completeness.

Proposition 4.2. *In the above notation,*

$$0 \to H^0(P_n, \mathbb{Q}) \xrightarrow{d^0} H^1(P_n, \mathbb{Q}) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H^n(P_n, \mathbb{Q}) \xrightarrow{d^n} 0 \qquad (4.1)$$

is an exact sequence of S_n-representations. Set $A_n^k := \mathrm{Im}(d^k) \subset H^{k+1}(P_n, \mathbb{Q})$. Hence we have an isomorphism of S_n-representations for each k,

$$H^k(P_n, \mathbb{Q}) \cong A_n^{k-1} \oplus A_n^k.$$

Proof. Arnol'd [1, Cor. 3] describes an additive basis $\mathcal{B}_k$ for $H^k(P_n, \mathbb{Q})$ comprised of all simple wedge products

$$\omega_{i_1,j_1} \wedge \cdots \wedge \omega_{i_k,j_k} \text{ such that } i_s < j_s \text{ for each } s, \text{ and } j_1 < j_2 < \ldots < j_k.$$

Let

$$U_k = \{\omega_{i_1,j_1} \wedge \cdots \wedge \omega_{i_k,j_k} \in \mathcal{B}_k : (i_s, j_s) \neq (n-1, n)\},$$

for $k > 0$ and $U_0 = \{1\}$. Then set

$$\mathcal{C}_k = U_k \cup \{\omega \wedge \tau : \omega \in U_{k-1}\}.$$

Claim. *$\mathcal{C}_k$ is a basis of $H^k(P_n, \mathbb{Q})$.*

For example, we have

$$\mathcal{C}_1 = \{\omega_{i,j} : (i,j) \neq (n-1, n)\} \cup \{\tau\},$$

which is clearly a basis for $H^1(P_n, \mathbb{Q})$.

To prove the claim, since $|\mathcal{B}_k| = |\mathcal{C}_k|$, it suffices to show $\mathcal{C}_k$ spans. Note that

$$\mathcal{B}_k = U_k \cup \{\omega \wedge \omega_{n-1,n} : \omega \in U_{k-1}\},$$

further reducing the problem to expressing $\omega \wedge \omega_{n-1,n}$ as a linear combination of $\mathcal{C}_k$ for each $\omega \in U_{k-1}$. Given $\omega = \omega_{i_1,j_1} \wedge \cdots \wedge \omega_{i_{k-1},j_{k-1}} \in U_{k-1}$, we use the relation

$$\omega_{i_s,j} \wedge \omega_{i,j} = \omega_{i_s,i} \wedge \omega_{i,j} - \omega_{i_s,i} \wedge \omega_{i_s,j}$$

to express $\omega \wedge \omega_{i,j}$ in terms of elements of U_k as follows:

$$\omega \wedge \omega_{i,j} = \begin{cases} \pm\omega_{i_1,j_1} \wedge \cdots \wedge \omega_{i_s,j_s} \wedge \omega_{i,j} \wedge \omega_{i_{s+1},j_{s+1}} \wedge \cdots \wedge \omega_{i_{k-1},j_{k-1}} \\ \qquad\qquad\qquad\qquad \text{for} \quad j_s < j < j_{s+1}, \\ \pm\omega_{i_1,j_1} \wedge \cdots \wedge (\omega_{i_s,i} \wedge \omega_{i,j} - \omega_{i_s,i} \wedge \omega_{i_s,j}) \wedge \cdots \wedge \omega_{i_{k-1},j_{k-1}} \\ \qquad\qquad\qquad\qquad \text{for} \quad j_s = j, i_s \neq i, \\ 0 \qquad\qquad\qquad\qquad \text{for} \quad (i_s, j_s) = (i, j). \end{cases}$$

The first and third cases are easily seen to belong in the span of U_k. Since $i_s, i < j$ and j does not occur twice as a largest subscript in ω, we see inductively that the second case also belongs in the span of U_k. Therefore, $\omega \wedge \tau = \omega \wedge \omega_{n-1,n} + \nu$, where ν is in the span of U_k. Hence $\omega \wedge \omega_{n-1,n} = \omega \wedge \tau - \nu$ is in the span of $\mathcal{C}_k$ and we conclude that $\mathcal{C}_k$ is a basis, proving the claim.

We now show the sequence (4.1) is exact. Suppose $\nu \in \ker(d^k)$. Express ν in the basis $\mathcal{C}_k$ as

$$\nu = \sum_{\omega \in U_k} a_\omega \, \omega + \sum_{\omega \in U_{k-1}} b_\omega \, \omega \wedge \tau.$$

Then

$$0 = d^k(\nu) = \nu \wedge \tau = \sum_{\omega \in U_k} a_\omega \, \omega \wedge \tau.$$

Since $\omega \wedge \tau$ is an element of the basis $\mathcal{C}_{k+1}$ for each $\omega \in U_k$, we have $a_\omega = 0$. Hence, $\nu = \mu \wedge \tau = d^{k-1}(\mu)$ where

$$\mu = \sum_{\omega \in U_{k-1}} b_\omega \, \omega,$$

so $\ker(d^k) = \mathrm{Im}(d^{k-1})$. □

Recall from Section 3.5 that the dimension of $H^k(P_n, \mathbb{Q})$ is given by an unsigned Stirling number of the first kind

$$\dim\left(H^k(P_n, \mathbb{Q})\right) = \begin{bmatrix} n \\ n-k \end{bmatrix},$$

where the unsigned Stirling numbers are determined by the identity $\prod_{k=0}^{n-1} (x+k) = \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} x^k$. The exact sequence in Proposition 4.2 shows the dimension of A_n^k is

$$\dim(A_n^k) = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} n \\ n-k+j \end{bmatrix}.$$

Table 4 gives values of $\dim(A_n^k)$ for small n and k; here $\dim(A_n^{n-1}) = 0$ for $n \geq 2$.

$n \setminus k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0
3	1	2	0	0	0	0	0	0
4	1	5	6	0	0	0	0	0
5	1	9	26	24	0	0	0	0
6	1	14	71	154	120	0	0	0
7	1	20	155	580	1044	720	0	0
8	1	27	295	1665	5104	8028	5040	0
9	1	35	511	4025	18424	48860	69264	40320

TABLE 4. $\dim(A_n^k)$

5. POLYNOMIAL SPLITTING MEASURES AND CHARACTERS

We now express the splitting measure coefficients $\alpha_n^k(C_\lambda)$ in terms of the character values $\chi_n^k(\lambda)$ where χ_n^k is the character of the S_n-representation A_n^k constructed in Proposition 4.2. As a corollary we deduce that the rescaled z-splitting measures are characters when $z = -\frac{1}{m}$ and virtual characters when $z = \frac{1}{m}$, generalizing results from [12].

5.1. **Expressing splitting measure coefficients by characters.** Recall,

$$\nu_{n,z}^*(C_\lambda) = \frac{N_\lambda(z)}{z^n - z^{n-1}} = \sum_{k=0}^{n-1} \alpha_n^k(C_\lambda)\left(\tfrac{1}{z}\right)^k.$$

We now express the splitting measure coefficient $\alpha_n^k(C_\lambda)$ in terms of the character value $\chi_n^k(\lambda)$.

Theorem 5.1. *Let $n \geq 2$ and λ be a partition of n, then*

$$\nu_{n,z}^*(C_\lambda) = \frac{1}{z_\lambda} \sum_{k=0}^{n-1} (-1)^k \chi_n^k(\lambda)\left(\tfrac{1}{z}\right)^k,$$

where χ_n^k is the character of the S_n-representation A_n^k defined in Proposition 4.2. Thus,

$$\alpha_n^k(C_\lambda) = \frac{1}{z_\lambda}(-1)^k \chi_n^k(\lambda).$$

Proof. In Theorem 3.2 we showed

$$N_\lambda(z) = \frac{1}{z_\lambda} \sum_{k=0}^{n} (-1)^k h_n^k(\lambda) z^{n-k},$$

where h_n^k is the character of $H^k(P_n, \mathbb{Q})$. The S_n-representations A_n^k were defined in Proposition 4.2 where we showed that

$$H^k(P_n, \mathbb{Q}) \cong A_n^{k-1} \oplus A_n^k. \tag{5.1}$$

Taking characters in (5.1) gives

$$h_n^k = \chi_n^{k-1} + \chi_n^k.$$

We compute

$$\begin{aligned}\frac{N_\lambda(z)}{z^n} &= \frac{1}{z_\lambda}\sum_{k=0}^{n}(-1)^k h_n^k(\lambda)\left(\tfrac{1}{z}\right)^k \\ &= \frac{1}{z_\lambda}\sum_{k=0}^{n}(-1)^k\left(\chi_n^{k-1}(\lambda)+\chi_n^k(\lambda)\right)\left(\tfrac{1}{z}\right)^k \\ &= \left(1-\tfrac{1}{z}\right)\frac{1}{z_\lambda}\sum_{k=0}^{n-1}(-1)^k\chi_n^k(\lambda)\left(\tfrac{1}{z}\right)^k.\end{aligned}$$

Dividing both sides by $\left(1-\frac{1}{z}\right)$ yields

$$\nu_{n,z}^*(C_\lambda) = \frac{N_\lambda(z)}{\left(1-\frac{1}{z}\right)z^n} = \frac{1}{z_\lambda}\sum_{k=0}^{n-1}(-1)^k\chi_n^k(\lambda)\left(\tfrac{1}{z}\right)^k.$$

Comparing coefficients in the two expressions for $\nu_{n,z}^*(C_\lambda)$ we find

$$\alpha_n^k(C_\lambda) = \frac{1}{z_\lambda}(-1)^k\chi_n^k(\lambda).$$

□

5.2. **Splitting measures for $z = \pm\frac{1}{m}$.** Representation-theoretic interpretations of the rescaled z-splitting measures for $z = \pm 1$ were studied in [12, Sec. 5]. Theorem 1.3 below generalizes those results to give representation-theoretic interpretations for $z = \pm\frac{1}{m}$ when $m \geq 1$ is an integer.

Theorem 5.2. *Let $n \geq 2$ and λ be a partition of n, then*
(1) For $z = -\frac{1}{m}$ with $m \geq 1$ an integer, we have

$$\nu_{n,-\frac{1}{m}}^*(C_\lambda) = \frac{1}{z_\lambda}\sum_{k=0}^{n-1}\chi_n^k(\lambda)m^k.$$

The function $z_\lambda\nu_{n,-\frac{1}{m}}^*(C_\lambda)$ is therefore the character of the S_n-representation

$$B_{n,m} = \bigoplus_{k=0}^{n-1}\left(A_n^k\right)^{\oplus m^k},$$

with dimension

$$\dim B_{n,m} = \prod_{j=2}^{n-1}(1+jm).$$

(2) For $z = \frac{1}{m}$ with $m \geq 1$ an integer, we have

$$\nu_{n,\frac{1}{m}}^*(C_\lambda) = \frac{1}{z_\lambda}\sum_{k=0}^{n-1}(-1)^k\chi_n^k(\lambda)m^k.$$

The function $z_\lambda \nu^*_{n,\frac{1}{m}}(C_\lambda)$ is a virtual character, the difference of characters of representations $B^+_{n,m}$ and $B^-_{n,m}$,

$$B^+_{n,m} \cong \bigoplus_{2j<n} \big(A_n^{2j}\big)^{\oplus m^{2j}} \qquad B^-_{n,m} \cong \bigoplus_{2j+1<n} \big(A_n^{2j+1}\big)^{\oplus m^{2j+1}}.$$

These representations have dimensions

$$\dim B^\pm_{n,m} = \frac{1}{2}\Big(\prod_{j=2}^{n-1}(1+jm) \pm \prod_{j=2}^{n-1}(1-jm)\Big)$$

respectively.

Proof. (1) The formula for the $(-\frac{1}{m})$-splitting measure follows by substituting $z = -\frac{1}{m}$ in Theorem 5.1. Arnol'd [1, Cor. 2] gives the Poincaré polynomial $p(w)$ of the pure braid group P_n as

$$p(w) = \prod_{j=1}^{n-1}(1+jw) = \sum_{k=0}^{n} h_n^k\big((1^n)\big)w^k.$$

On the other hand, by Theorem 3.2 we have

$$n!(-1)^n w^n N_{(1^n)}(-w^{-1}) = \sum_{k=0}^{n} h_n^k\big((1^n)\big)w^k. \tag{5.2}$$

Dividing (5.2) by $1+w$ we have

$$\prod_{j=2}^{n-1}(1+jw) = n!(-1)^n w^n \frac{N_{(1^n)}(-w^{-1})}{1+w} = \sum_{k=0}^{n-1} \chi_n^k\big((1^n)\big)w^k. \tag{5.3}$$

Substituting $w = m$ gives the dimension formula.

(2) Substituting $z = \frac{1}{m}$ in Theorem 5.1 gives the formula for the $(\frac{1}{m})$-splitting measure. Separating the even and odd parts we have

$$z_\lambda \nu^*_{n,\frac{1}{m}}(C_\lambda) = \sum_{2j<n} \chi_n^{2j}(\lambda)m^{2j} - \sum_{2j+1<n} \chi_n^{2j+1}(\lambda)m^{2j+1}.$$

Hence $z_\lambda \nu^*_{n,\frac{1}{m}}(C_\lambda) = \chi^+_{n,m}(\lambda) - \chi^-_{n,m}(\lambda)$, where $\chi^\pm_{n,m}$ are characters of $B^\pm_{n,m}$ respectively. The dimension formulas follow from decomposing (5.3) into even and odd parts. □

Remark. Other results in [12, Theorems 3.2, 5.2 and 6.1] determine the values of the rescaled splitting measures for $z = \pm 1$, showing they are supported on remarkably few conjugacy classes; for $z = 1$ these were the Springer regular elements of S_n. Theorem 1.3 does not explain the small support of the characters for $z = \pm 1$. The characters h_n^k and χ_n^k have large support in general, hence cancellation must occur to explain the small support. It would be interesting to account for this phenomenon.

5.3. **Cohomology of the pure braid group and the regular representation.** We use Theorem 5.1 together with the splitting measure values at $z = -1$ computed in [12] to determine a relation between the S_n-representation structure of the pure braid group cohomology and the regular representation of S_n. Let A_n^k be the S_n-subrepresentation constructed in Proposition 4.2, and define the S_n-representation

$$A_n := \bigoplus_{k=0}^{n-1} A_n^k.$$

Theorem 5.3. *Let $\mathbf{1}_n$, $\mathbf{Sgn}_n$, and $\mathbb{Q}[S_n]$ denote the trivial, sign, and regular representations of S_n respectively. Then there are isomorphisms of S_n-representations,*

$$\bigoplus_{k=0}^{n} H^k(P_n, \mathbb{Q}) \otimes \mathbf{Sgn}_n^{\otimes k} \cong \mathbb{Q}[S_n].$$

and

$$A_n \otimes \big(\mathbf{1}_n \oplus \mathbf{Sgn}_n^{\otimes k}\big) \cong \mathbb{Q}[S_n].$$

Proof. We showed in Proposition 4.2 that $H^k(P_n, \mathbb{Q}) \cong A_n^{k-1} \oplus A_n^k$, with $A_n^{-1} = A_n^n = 0$. Therefore, summing over $0 \leq k \leq n$,

$$A_n \cong \bigoplus_{k \text{ even}} H^k(P_n, \mathbb{Q}) \cong \bigoplus_{k \text{ odd}} H^k(P_n, \mathbb{Q}).$$

Since $\mathbf{Sgn}_n^{\otimes 2} \cong \mathbf{1}_n$, we have

$$\begin{aligned}\bigoplus_{k=0}^{n} H^k(P_n, \mathbb{Q}) \otimes \mathbf{Sgn}_n^{\otimes k} &\cong \Big(\bigoplus_{k \text{ even}} H^k(P_n, \mathbb{Q}) \otimes \mathbf{1}_n\Big) \oplus \Big(\bigoplus_{k \text{ odd}} H^k(P_n, \mathbb{Q}) \otimes \mathbf{Sgn}_n\Big) \\ &\cong (A_n \otimes \mathbf{1}_n) \oplus (A_n \otimes \mathbf{Sgn}_n) \\ &\cong A_n \otimes (\mathbf{1}_n \oplus \mathbf{Sgn}_n).\end{aligned}$$

If χ_n is the character of A_n, then it follows from Theorem 1.3 that

$$\chi_n(\lambda) = \sum_{k=0}^{n-1} \chi_n^k(\lambda) = z_\lambda \nu_{n,-1}^*(C_\lambda),$$

so the values of χ_n are given by the rescaled (-1)-splitting measure.

Theorem 6.1 of [12] shows

$$\nu_{n,-1}^*(C_\lambda) = \begin{cases} \frac{1}{2} & \lambda = (1^n) \text{ or } (1^{n-2}\,2), \\ 0 & \text{otherwise.} \end{cases}$$

Now let $\rho = \chi_n \cdot (1_n + \mathrm{sgn}_n)$ be the character of $A_n \otimes (\mathbf{1}_n \oplus \mathbf{Sgn}_n)$. If $\lambda = (1^n)$, we compute

$$\rho(\lambda) = \chi_n(\lambda)\big(1 + \mathrm{sgn}_n(\lambda)\big) = n!\nu_{n,-1}^*(C_\lambda)(2) = n!.$$

If $\lambda = (1^{n-2}\,2)$, then $\big(1 + \mathrm{sgn}_n(\lambda)\big) = 0$, hence $\rho(\lambda) = 0$. If λ is any other partition, then $\nu_{n,-1}^*(C_\lambda) = 0$, hence $\rho(\lambda) = 0$. Therefore ρ agrees with the character of the regular representation, proving

$$\bigoplus_{k=0}^{n} H^k(P_n, \mathbb{Q}) \otimes \mathbf{Sgn}_n^{\otimes k} \cong A_n \otimes (\mathbf{1}_n \oplus \mathbf{Sgn}_n) \cong \mathbb{Q}[S_n].$$

□

6. Representation Stability

In previous sections we expressed the coefficients of cycle polynomials and splitting measures in terms of characters of S_n-representations. These sequences of representations are known to exhibit a phenomenon described by Church and Farb [6] as representation stability.

6.1. **Stability of S_n-representations.** The irreducible representations of S_n are naturally parametrized by partitions λ of n. Let $\mathcal{S}^\lambda$ be an irreducible representation of S_n corresponding to λ. Say V_n is a sequence of finite dimensional S_n-representations, and let n_0 be a fixed positive integer. Then V_{n_0} has an irreducible decomposition

$$V_{n_0} \cong \bigoplus_{|\lambda|=n_0} \left(\mathcal{S}^\lambda\right)^{\oplus e_\lambda}.$$

Following Church and Farb, we say the sequence V_n *stabilizes at* n_0 if for each $n \geq n_0$ we have

$$V_n \cong \bigoplus_{|\lambda|=n_0} \left(\mathcal{S}^{\lambda+(n-n_0)}\right)^{\oplus e_\lambda},$$

where for a non-negative integer m the partition $\lambda+m$ is defined as $[\lambda_1+m, \lambda_2, \ldots, \lambda_\ell]$ when $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$. In other words, the sequence V_n stabilizes at n_0 if all the irreducible decompositions of subsequent V_n are determined by the decomposition of V_{n_0}. It *stabilizes sharply at* n_0 if n_0 is the least integer with this property.

If a sequence V_n of S_n-representations stabilizes, then the characters for V_n have a uniform description for all sufficiently large n given by a *polynomial character* χ_P where $P(x) \in \mathbb{Q}[x_j : j \geq 1]$ is a character polynomial. If λ is a partition, then

$$\chi_P(\lambda) := P\big(m_1(\lambda), m_2(\lambda), \ldots\big),$$

noting that $m_j(\lambda) = 0$ for all but finitely many j. If m is the largest index of a variable x_m occurring in $P(x)$, then χ_P is determined by $m_j(\lambda)$ for $j \leq m$.

Church and Farb [6] introduced the notion of representation stability to describe a collection of closely related and frequently observed phenomenon. The pure braid group cohomology $H^k(P_n, \mathbb{Q})$ provided one of their initial examples of representation stability. Recently Hersh and Reiner [11] studied the representation stability of the cohomology of configuration space of n points in $\mathbb{R}^d$, which includes the pure braid group cohomology as the case $d = 2$. Their results imply that for fixed k, both sequences $H^k(P_n, \mathbb{Q})$ and A_n^k stabilize sharply at $n_0 = 3k + 1$, as we state in Theorem 6.3 below.

We illustrate the stability phenomenon through the irreducible decompositions of $H^k(P_n, \mathbb{Q})$ and A_n^k for $k = 1$ ($n_0 = 4$) in Table 5 and for $k = 2$ ($n_0 = 7$) in Table 6 and Table 7.

n	$\dim H^1$	$H^1(P_n, \mathbb{Q})$	$\dim A_n^1$	A_n^1
2	1	$[2]$	0	0
3	3	$[3] \oplus [2,1]$	2	$[2,1]$
4	6	$[4] \oplus [3,1] \oplus [2,2]$	5	$[3,1] \oplus [2,2]$
5	10	$[5] \oplus [4,1] \oplus [3,2]$	9	$[4,1] \oplus [3,2]$
n	$\left[{n \atop n-1}\right]$	$[n] \oplus [n-1,1] \oplus [n-2,2]$	$\left[{n \atop n-1}\right] - 1$	$[n-1,1] \oplus [n-2,2]$

TABLE 5. Irreducible decompositions for $H^1(P_n, \mathbb{Q})$ and A_n^1. Here λ abbreviates the irreducible representation $\mathcal{S}^\lambda$.

n	$\dim H^2$	$H^2(P_n, \mathbb{Q})$
3	2	$[2,1]$
4	11	$2[3,1] \oplus [2,2] \oplus [2,1,1]$
5	35	$2[4,1] \oplus 2[3,2] \oplus 2[3,1,1] \oplus [2,2,1]$
6	85	$2[5,1] \oplus 2[4,2] \oplus 2[4,1,1] \oplus [3,3] \oplus 2[3,2,1]$
7	175	$2[6,1] \oplus 2[5,2] \oplus 2[5,1,1] \oplus [4,3] \oplus 2[4,2,1] \oplus [4,3,1]$
8	322	$2[7,1] \oplus 2[6,2] \oplus 2[6,1,1] \oplus [5,3] \oplus 2[5,2,1] \oplus [4,3,1]$
n	$\left[{n \atop n-2}\right]$	$2[n-1,1] \oplus 2[n-2,2] \oplus 2[n-2,1,1] \oplus [n-3,3]$ $\oplus 2[n-3,2,1] \oplus [n-4,3,1]$

TABLE 6. Irreducible decomposition for $H^2(P_n, \mathbb{Q})$

n	$\dim A_n^2$	A_n^2
3	0	0
4	6	$[3,1] \oplus [2,1,1]$
5	26	$[4,1] \oplus [3,2] \oplus 2[3,1,1] \oplus [2,2,1]$
6	71	$[5,1] \oplus [4,2] \oplus 2[4,1,1] \oplus [3,3] \oplus 2[3,2,1]$
7	155	$[6,1] \oplus [5,2] \oplus 2[5,1,1] \oplus [4,3] \oplus 2[4,2,1] \oplus [3,3,1]$
8	295	$[7,1] \oplus [6,2] \oplus 2[6,1,1] \oplus [5,3] \oplus 2[5,2,1] \oplus [4,3,1]$
n	$\left[{n \atop n-2}\right] - \left[{n \atop n-1}\right] + 1$	$[n-1,1] \oplus [n-2,2] \oplus 2[n-2,1,1] \oplus [n-3,3]$ $\oplus 2[n-3,2,1] \oplus [n-4,3,1]$

TABLE 7. Irreducible decomposition for A_n^2

The associated character polynomials for $H^1(P_n, \mathbb{Q})$ and $H^2(P_n, \mathbb{Q})$ were given in Example 3.7. In general, given a stable sequence V_n of S_n-representations, it appears difficult to determine when it sharply stabilizes, and to determine the associated character polynomial [9, Problem 4.7].

6.2. **Representation stability for $H^k(P_n, \mathbb{Q})$ and A_n^k.** Let Π_n denote the collection of partitions of a set with n elements, partially ordered by refinement (see Stanley [19, Example 3.10.4]).

If $0 = C^0, C^1, C^2, \ldots$ is any sequence of *semisimple modules* with submodules $B^k \subseteq C^k$, then isomorphisms

$$C^k \cong B^{k-1} \oplus B^k$$

for each k determine the B^k up to isomorphism. Finite dimensional S_n-representations are semisimple by Maschke's theorem, hence

$$H^k(P_n, \mathbb{Q}) \cong A_n^{k-1} \oplus A_n^k \tag{6.1}$$

determines A_n^k up to isomorphism.

Hersh and Reiner [11, Sec. 2] describe two other sequences of S_n-representations giving direct sum decompositions of $H^k(P_n, \mathbb{Q})$ coming from the Whitney and simplicial homology of the lattice Π_n.

Proposition 6.1. *(1) There is an isomorphism of S_n-representations*

$$H^k(P_n, \mathbb{Q}) \cong WH_k(\Pi_n), \tag{6.2}$$

where $WH_k(\Pi_n)$ is the kth Whitney homology of the lattice Π_n.

(2) There is an isomorphism of S_n-representations

$$WH_k(\Pi_n) \cong \beta_{[k-1]}(\Pi_n) \oplus \beta_{[k]}(\Pi_n)$$

where $\beta_{[k]}(\Pi_n)$ is the $[k] = \{1, 2, ..., k\}$-rank selected homology of the lattice Π_n.

(3) There is an isomorphism of S_n-representations

$$\beta_{[k]}(\Pi_n) \cong \widetilde{H}_{k-1}(\Pi_n^k),$$

where Π_n^k is the sub-poset of $\lambda \in \Pi_n$ with $|\lambda| - \ell(\lambda) \leq k$ and $\widetilde{H}_{k-1}(\Pi_n^k)$ denotes its reduced simplicial homology.

Proof. (1) This result is due to Sundaram and Welker [21, Theorem 4.4 (iii)], cf. [11, Thm. 2.11, Sec. 2.3]. (See [11, Sec. 2.4] for more on the Whitney homology of Π_n.)

(2) Sundaram [20, Prop. 1.9] decomposes $WH_k(\Pi_n)$ as

$$WH_k(\Pi_n) \cong \beta_{[k-1]}(\Pi_n) \oplus \beta_{[k]}(\Pi_n), \tag{6.3}$$

where $[k] = \{1, 2, \ldots, k\}$ and $\beta_{[k]}(\Pi_n)$ is the $[k]$-*rank selected homology* of the lattice Π_n [11, Prop. 2.17].

(3) Because the lattice Π_n is *Cohen-Macaulay*, Hersh and Reiner note [11, Sec. 2.5] the isomorphism

$$\beta_{[k]}(\Pi_n) \cong \widetilde{H}_{k-1}(\Pi_n^k), \tag{6.4}$$

where Π_n^k is the sub-poset of $\lambda \in \Pi_n$ with $|\lambda| - \ell(\lambda) \leq k$ and $\widetilde{H}_{k-1}(\Pi_n^k)$ is its *reduced simplicial homology*. $\square$

The following proposition relates A_n^k, $\beta_{[k]}(\Pi_n)$, and $\widetilde{H}_{k-1}(\Pi_n^k)$ using (6.1).

Proposition 6.2. *Let A_n^k be as defined in Prop. 4.2, Π_n be the lattice of partitions of an n-element set, and $\Pi_n^k \subseteq \Pi_n$ the sub-poset comprised of $\lambda \in \Pi_n$ with $|\lambda| - \ell(\lambda) \leq k$. Then we have the following isomorphisms of S_n-representations*

$$A_n^k \cong \beta_{[k]}(\Pi_n) \cong \widetilde{H}_{k-1}\big(\Pi_n^k\big).$$

Proof. The isomorphisms (6.2) and (6.3) in Proposition 6.1 give the direct sum decompositions

$$H^k(P_n, \mathbb{Q}) \cong \beta_{[k-1]}(\Pi_n) \oplus \beta_{[k]}(\Pi_n)$$

for $0 \leq k \leq n$. By (6.1) we have that

$$H^k(P_n, \mathbb{Q}) \cong A_n^{k-1} \oplus A_n^k.$$

Since for $k = 0$,

$$\beta_{[-1]}(\Pi_n) \cong A_n^{-1} = \{0\},$$

we obtain by induction on $k \geq 1$ that

$$A_n^k \cong \beta_{[k]}(\Pi_n)$$

Combining this isomorphism with (6.4) finishes the proof. $\square$

Hersh and Reiner [11] prove sharp stability results for various sequences of S_n-representations related to configuration spaces. We conclude this section by expressing their stability results in our context.

Theorem 6.3 (Stability for splitting measure coefficients)**.** *For each $k \geq 1$,*

(1) The sequence of S_n-representations $H^k(P_n, \mathbb{Q})$, with characters h_n^k, stabilizes sharply at $n_0 = 3k + 1$.

(2) The sequence of S_n-representations A_n^k with characters χ_n^k, stabilizes sharply at $n_0 = 3k + 1$.

Proof. (1) This sharp stability result is the special case $d = 2$ of [11, Theorem 1.1].

(2) Corollary 4.4 of [11] shows that the sequence $\beta_{[k]}(\Pi_n)$ of S_n-representations stabilizes sharply at $n_0 = 3k + 1$. Proposition 6.2 gives the isomorphism $A_n^k \cong \beta_{[k]}(\Pi_n)$ and the result follows. $\square$

7. Continuity of the normalized splitting measure

In this section we introduce the *normalized splitting measure* and show how it manifests the stability of the splitting measure coefficients through a "continuity property" with respect to certain natural limits defined below.

7.1. **Normalized splitting measures.**

Definition 7.1. The *normalized splitting measure* $\nu_w : \mathrm{Par} \to \mathbb{Z}[w]$ is defined on a partition $\lambda \in \mathrm{Par}$ with $|\lambda| = n \geq 2$ by

$$\nu_w(\lambda) := z_\lambda \frac{w^n N_\lambda(w^{-1})}{1 - w}$$

and $\nu_w(\lambda) = 1$ for the unique partition λ of 1.

The normalized splitting measure $\nu_w(\lambda)$ is a polynomial in w with integer coefficients. The normalized splitting measure relates to the z-splitting measure by

$$\nu_w(\lambda) = z_\lambda \nu^*_{|\lambda|, \frac{1}{w}}(C_\lambda).$$

All $\nu_w(\lambda)$ have constant term 1, corresponding to the property that the z-splitting measure is the uniform distribution on S_n at $z = \infty$. Theorem 5.1 and (7.1) give the following expression for the normalized splitting measure $\nu_w(\lambda)$ in terms of the character values $\chi_n^k(\lambda)$ when $|\lambda| = n$,

$$\nu_w(\lambda) = \sum_{k=0}^{n-1} (-1)^k \chi_n^k(\lambda) w^k.$$

In Tables 8 and 9 we give values of $\nu^*_{n,z}(C_\lambda)$ and of the normalized splitting measure $\nu_w(\lambda)$ for $n = 4$ and $n = 5$, extending Tables 1 and 2.

λ	c_λ	z_λ	$\nu^*_{4,z}(C_\lambda)$	$\nu_w(\lambda)$
$[1,1,1,1]$	1	24	$\frac{1}{24}\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)$	$1 - 5w + 6w^2$
$[2,1,1]$	6	4	$\frac{1}{4}\left(1 - \frac{1}{z}\right)$	$1 - w$
$[2,2]$	3	8	$\frac{1}{8}\left(1 - \frac{1}{z} - \frac{2}{z^2}\right)$	$1 - w - 2w^2$
$[3,1]$	8	3	$\frac{1}{3}\left(1 + \frac{1}{z}\right)$	$1 + w$
$[4]$	6	4	$\frac{1}{4}\left(1 + \frac{1}{z}\right)$	$1 + w$

TABLE 8. Values of the splitting measure $\nu^*_{4,z}(C_\lambda)$ and Normalized splitting measure $\nu_w(\lambda)$ on partitions λ of for $n = 4$.

λ	c_λ	z_λ	$\nu^*_{5,z}(C_\lambda)$	$\nu_w(\lambda)$
$[1,1,1,1,1]$	1	120	$\frac{1}{120}\left(1 - \frac{9}{z} + \frac{26}{z^2} - \frac{24}{z^3}\right)$	$1 - 9w + 26w^2 - 24w^3$
$[2,1,1,1]$	10	12	$\frac{1}{12}\left(1 - \frac{3}{z} + \frac{2}{z^2}\right)$	$1 - 3w + 2w^2$
$[2,2,1]$	15	8	$\frac{1}{8}\left(1 - \frac{1}{z} - \frac{2}{z^2}\right)$	$1 - w - 2w^2$
$[3,1,1]$	20	6	$\frac{1}{6}\left(1 - \frac{1}{z^2}\right)$	$1 - w^2$
$[3,2]$	20	6	$\frac{1}{6}\left(1 - \frac{1}{z^2}\right)$	$1 - w^2$
$[4,1]$	30	4	$\frac{1}{4}\left(1 + \frac{1}{z}\right)$	$1 + w$
$[5]$	24	5	$\frac{1}{5}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}\right)$	$1 + w + w^2 + w^3$

TABLE 9. Values of the splitting measure $\nu^*_{5,z}(C_\lambda)$ and normalized splitting measure values $\nu_w(\lambda)$ on partitions λ of $n = 5$.

7.2. **Stable sequences of partitions.** Viewing a partition λ as a non-increasing, eventually zero sequence of non-negative integers, we let λ_i denote the ith largest part of λ. We identify partitions with their Young diagrams written in British style. Given two partitions $\lambda^{(1)}, \lambda^{(2)}$, we say $\lambda^{(1)} \subseteq \lambda^{(2)}$ if $\lambda_i^{(1)} \leq \lambda_i^{(2)}$ for each i, or more visually, if the Young diagram of $\lambda^{(1)}$ fits inside that of $\lambda^{(2)}$. A *nested sequence* of partitions $\lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \lambda^{(3)} \subseteq \ldots$ is any infinite chain.

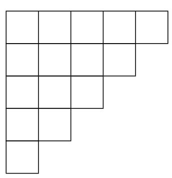
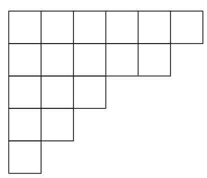
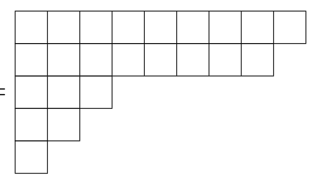
Definition 7.2. (1) We call a nested sequence of partitions $\big(\lambda^{(k)}\big)$ a *stable sequence* if the length sequence $\ell\big(\lambda^{(k)}\big)$ is bounded.

(2) Given a stable sequence $\big(\lambda^{(k)}\big)$, let $s \geq 1$ be minimal such that $\big(\lambda_s^{(k)}\big)$ is eventually constant, then we define the *stable limit* of $\big(\lambda^{(k)}\big)$ to be the partition

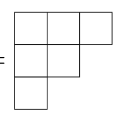
$$\operatorname*{stablim}_{k\to\infty} \lambda^{(k)} := \lim_{k\to\infty} [\lambda_s^{(k)}, \lambda_{s+1}^{(k)}, \lambda_{s+2}^{(k)}, \ldots],$$

where the limit on the right hand side is taken componentwise.

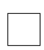
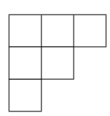
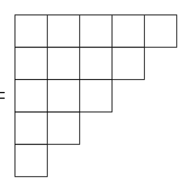
Example 7.3. (1) The sequence of partitions $\lambda^{(k)} = [k+4, k+3, 3, 2, 1]$ is a stable sequence, since $\ell\big(\lambda^{(k)}\big) = 5$ for each k.

$\lambda^{(1)} =$  $\lambda^{(2)} =$  $\lambda^{(5)} =$ 

The smallest s for which $\big(\lambda_s^{(k)}\big)$ is bounded is $s = 3$. The stable limit of $\big(\lambda^{(k)}\big)$ is

$$\operatorname*{stablim}_{k\to\infty} \lambda^{(k)} = [3, 2, 1] =$$ 

(2) The staircase sequence $\mu^{(k)} = [k, k-1, k-2, \ldots, 1]$ is not a stable sequence since $\ell\big(\mu^{(k)}\big) = k$ is unbounded.

$\mu^{(1)} =$  $\mu^{(3)} =$  $\mu^{(5)} =$ 

Remark. Stable sequences of partitions can have sequences of blocks going to infinity at different rates, e.g. $\mu^{(k)} = [k^2 + 4, k + 3, 3, 2, 1]$ is a stable sequence having the same stable limit as $\lambda^{(k)} = [k + 4, k + 3, 3, 2, 1]$.

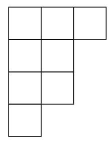
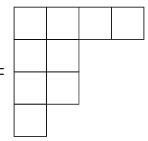
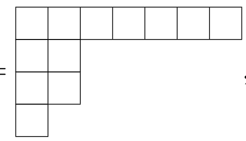
7.3. **Continuity with respect to stable limits.** Recall from Section 7.1 that the normalized splitting measure ν_w defined by

$$\nu_w(\lambda) = z_\lambda \frac{w^n N_\lambda(w^{-1})}{1 - w},$$

where $\lambda \in \mathrm{Par}$ is a partition.

We show that the normalized splitting measure ν_w is *continuous with respect to stable limits*. Before stating the precise result we demonstrate with an example.

Example 7.4. Consider the stable sequence $\lambda^{(k)} = [k+2, 2, 2, 1]$.

$\lambda^{(1)} =$  $\lambda^{(2)} =$  $\lambda^{(5)} =$  ,

We compute values of $\nu_w\big(\lambda^{(k)}\big)$ in Table 10.

$\lambda^{(k)}$	$\nu_w(\lambda^{(k)})$
$\lambda^{(1)}$	$1 - w - 3w^2 + O\big(w^3\big)$
$\lambda^{(10)}$	$1 - w - 2w^2 + O\big(w^6\big)$
$\lambda^{(100)}$	$1 - w - 2w^2 + O\big(w^{51}\big)$
$\lambda^{(1000)}$	$1 - w - 2w^2 + O\big(w^{501}\big)$

TABLE 10. Values of $\nu_w\big(\lambda^{(k)}\big)$ for $\lambda^{(k)} = [k+2, 2, 2, 1]$.

The sequence $\nu_w(\lambda^{(k)})$ appears to converge coefficientwise to the limit $1 - w - 2w^2$. This convergence can be viewed in the formal power series ring $\mathbb{Z}[[w]]$, i.e. the w-adic completion of $\mathbb{Z}[w]$. Observe that $\lambda := \mathrm{stablim}_{k\to\infty} \lambda^{(k)} = [2, 2, 1]$ has normalized measure $\nu_w(\lambda) = 1 - w - 2w^2$. In other words,

$$\lim_{k\to\infty} \nu_w\big(\lambda^{(k)}\big) = \nu_w\big(\operatorname*{stablim}_{k\to\infty} \lambda^{(k)}\big), \tag{7.1}$$

where the limit on the left hand side is taken coefficientwise in $\mathbb{Z}[w]$ and the limit on the right hand side is the stable limit of the sequence $\big(\lambda^{(k)}\big)$ of partitions.

Theorem 7.5 (Continuity of ν_w with respect to stable limits)**.** *Suppose $\big(\lambda^{(k)}\big)$ is a stable sequence of partitions, then*

$$\lim_{k\to\infty} \nu_w\big(\lambda^{(k)}\big) = \nu_w\big(\operatorname*{stablim}_{k\to\infty} \lambda^{(k)}\big),$$

where the limit on the left hand side is taken coefficientwise in $\mathbb{Z}[w]$ and the limit on the right hand side is the stable limit of a sequence of partitions.

Proof. All limits of sequences of polynomials in $\mathbb{Z}[w]$ are taken coefficientwise, i.e. are limits in the formal power series ring $\mathbb{Z}[[w]]$. We first claim that

$$\lim_{j\to\infty} jw^j M_j\big(w^{-1}\big) = 1. \tag{7.2}$$

Recalling the definition of the jth necklace polynomial, we compute

$$jw^j M_j\big(w^{-1}\big) = \sum_{d|j} \mu(d) w^{j\left(1-\frac{1}{d}\right)} = 1 + \sum_{\substack{d|j\\ d\neq 1}} \mu(d) w^{j\left(1-\frac{1}{d}\right)}.$$

Since $d \geq 2$, $\left(1 - \frac{1}{d}\right) \geq \frac{1}{2}$, and thus

$$w^{j/2} \mid jw^j M_j\big(w^{-1}\big) - 1, \tag{7.3}$$

giving the limit (7.2). Next we claim that for any integer $m \geq 1$,

$$\lim_{j\to\infty} j^m m! w^{jm} \binom{M_j\big(w^{-1}\big)}{m} = 1. \tag{7.4}$$

This follows from (7.2) after writing

$$j^m m! w^{jm} \binom{M_j\big(w^{-1}\big)}{m} = \prod_{k=0}^{m-1} \big(jw^j M_j\big(w^{-1}\big) - jw^j k\big),$$

and taking the limit of each factor.

If $\big(\lambda^{(k)}\big)$ is a stable sequence with s minimal such that $\big(\lambda_s^{(k)}\big)$ is eventually constant, set $\lambda := \operatorname{stablim}_{k\to\infty} \lambda^{(k)}$. Then $\lambda_s := \lim_{k\to\infty} \lambda_s^{(k)}$ is the largest part of λ. We compute

$$(1-w) \lim_{k\to\infty} \nu_w\big(\lambda^{(k)}\big) = \prod_{j\geq 1} \lim_{k\to\infty} j^{m_j(\lambda_k)} m_j(\lambda_k)! w^{jm_j(\lambda_k)} \binom{M_j\big(w^{-1}\big)}{m_j(\lambda^{(k)})}. \tag{7.5}$$

For any $j \leq \lambda_s$,

$$\lim_{k\to\infty} m_j(\lambda_k) = m_j(\lambda),$$

hence the the limit of the factors in (7.5) stabilizes. If $j > \lambda_s$ then $m_j(\lambda_k) = 0$ for all sufficiently large k, hence these factors in (7.5) all tend to 1 by (7.3).

Therefore

$$\begin{aligned} \lim_{n\to\infty} \nu_w\big(\lambda^{(k)}\big) &= \frac{1}{1-w} \prod_{1\leq j\leq \lambda_s} j^{m_j(\lambda)} m_j(\lambda)! w^{jm_j(\lambda)} \binom{M_j\big(w^{-1}\big)}{m_j(\lambda)} \\ &= \nu_w\big(\operatorname*{stablim}_{n\to\infty} \lambda^{(k)}\big). \end{aligned}$$

□

Acknowledgments. We thank Richard Stanley for raising a question about the relation of the braid group cohomology to the regular representation, answered by Theorem 1.3. We thank Philip Tosteson and John Wiltshire-Gordon for helpful conversations.

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DEPT. OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109-1043,
E-mail address: `tghyde@umich.edu`

DEPT. OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109-1043,
E-mail address: `lagarias@umich.edu`